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This report documents a method for using satellite data to compute a value for dilution of precision in geodetic positioning. The procedures described are for users receiving satellite information from almanac data, and the expressions defined for dilution of precision are for users at sea level, measuring receiver position in Earth-fixed, geodetic coordinates.

FOREWORD

This document was prepared and written under the authority of the Defense Mapping Agency, Washington D.C., 20305.

Acknowledgements are made to Stanley L. Meyerhoff, of the Space Flight Sciences Branch K13, Naval Surface Weapons Center (NSWC), for his invaluable support in the preparation of this report.

This report has been reviewed by J. N. Blanton, Head, Space Flight Sciences Branch; and C. W. Duke, Head, Space and Surface Systems Division.

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INTRODUCTION

Certain users of the Global Positioning System (GPS) use the geodetic coordinate system to define their receiver position. Satellite positions, computed from either ephemeris or almanac data, are best stated, however, in terms of Earth-centered, Earth-fixed (ECEF) cartesian coordinates. This presents a problem when computing values for a dilution of precision in position—the measure by which the four satellites most suitable for range measurements are determined. (DOP)

Before receiving transmissions from GPS satellites and using them to determine precise receiver position, the user must determine which four satellites are in the configuration that will maximize system accuracy. This is accomplished by computing a ~~dilution of precision~~ DOP value for all groups of four satellites that are in view. For GPS users who measure their receiver position in geodetic coordinates, this value should reflect the use of these coordinates, rather than ECEF cartesian coordinates.

This report outlines five algorithms, all used in the process of computing a ~~dilution of precision~~ DOP in position. The method to compute satellite position is described assuming the user is using almanac data, and the dilution of precision in position term is defined in terms of geodetic coordinates. (PDOP)

The basic procedure, as outlined by the algorithms, is as follows

Use almanac data to find satellite position at time (t)

Convert receiver position estimate from geodetic coordinates to ECEF cartesian coordinates

Establish whether a satellite is healthy and in view

Compute covariance matrixes for all groups of four satellites that are healthy and in view

Rotate each covariance matrix to be in terms of geodetic coordinates, and compute a dilution of precision in position

The group of four satellites that yields the smallest dilution of precision in position is the group to use when making range measurements from satellite transmissions.

Each of the five algorithms details how to accomplish one of the tasks previously stated. The items that are needed as input to the routines are listed and defined first. The steps and equations required to produce the desired output follows. The algorithms are presented in a straightforward manner, and computer code could be written from them.*

The appendixes provide a more thorough explanation of the equations and procedures detailed in the five algorithms. The collection of material found in the appendixes should prove to be as useful as the algorithms themselves.* *

None of the material in this report is developed for the first time here, but as far as it is known, it has never been assembled as in this report. The usefulness of having this information combined into one document was a major reason for assembling this report.

PROCEDURES

FINDING SATELLITE POSITION FROM ALMANAC DATA[1]

The user obtains the following input data from the almanac

ISAT	Satellite tracker number
WNA	Reference week of almanac data
e	Eccentricity
t_{oa}	Reference time of almanac data (s)
i_o	Inclination angle (0.30 semicircles)
δ_i	Correction to inclination (sc)
$\dot{\Omega}$	Rate of right ascension (sc/s)
$(a_e)^{1/2}$	Square root of semimajor axis length (m)

* These algorithms, or sections of these algorithms, already exist as code or in program-design documentation. These were assembled from Reference 1.

* * Nearly all the derivations collected in the appendixes can be found in either Reference 1 or 2.

Ω_o Right ascension (sc)

ω Argument of perigee (sc)

M_o Mean anomoly (sc)

The user provides the following input data

t Time at which position is to be known (s)

WN Week number at which position is to be known

μ_g WGS-72 value for the Earth's gravitational constant

$\dot{\Omega}_e$ WGS 72 value for the Earth's rotation rate

The user receives the following output data

x_s, y_s, z_s Satellite position

E Eccentric anomoly

Define semimajor axis length, a_e, mean motion, N, time difference (positive or negative) between t and t_{oa}, Δt, and a new mean anomoly, M.*

$$a_e = (a_e^{1/2})^2$$

$$N_1 = N_o = (\mu_g / a_e^3)^{1/2}$$

$$\Delta t = t + 604{,}800\,(WN - WNA) - t_{oa}$$

$$M = E = M_o + N_1 \cdot \Delta t$$

Solve for eccentric anomoly by repeating the next two lines eight times or until tau (τ) becomes less than or equal to 1×10^{-10}.

$$\tau = E - e \sin E - M$$

$$E = M + e \sin E$$

---

* See Appendix A for derivations of some of these equations. This routine is part of Reference 1; it is also contained in Reference 3.

Solve for true anomoly, ν, corrected radius, r, and corrected inclination i.

$$\sin \nu = (1 - e^2)^{1/2} \sin E / (1 - e \cos E)$$

$$\cos \nu = (\cos E - e) / (1 - e \cos E)$$

$$r = a_e (1 - e \cos E)$$

$$i = i_o + \delta_i$$

Find the corrected longitude of the ascending node and the position of the satellite in the orbital plane. (Let $u = \nu + \omega$)

$$\Omega = \Omega_o + \dot{\Omega} \cdot \Delta t - \dot{\Omega}_e (\Delta t + t_{oa})$$

$$x'_s = r \cdot \cos u = r \cdot [\cos \nu \cos \omega - \sin \nu \sin \omega]$$

$$y'_s = r \cdot \sin u = r \cdot [\sin \nu \cos \omega + \cos \nu \sin \omega]$$

Solve for position in the ECEF cartesian frame.

$$x_s = x'_s \cos \Omega - y'_s \cos i \sin \Omega$$

$$y_s = x'_s \sin \Omega + y'_s \cos i \cos \Omega$$

$$z_s = y'_s \sin i$$

CONVERTING RECEIVER POSITION FROM GEODETIC COORDINATES TO ECEF CARTESIAN COORDINATES[1]

The user inputs the following data

ϕ, λ, h   Geodetic latitude, longitude, and height of receiver

a_e Earth's semimajor axis length

e   Earth's eccentricity

The user obtains output data in the form of the x, y, and z receiver coordinates (Figure 1).

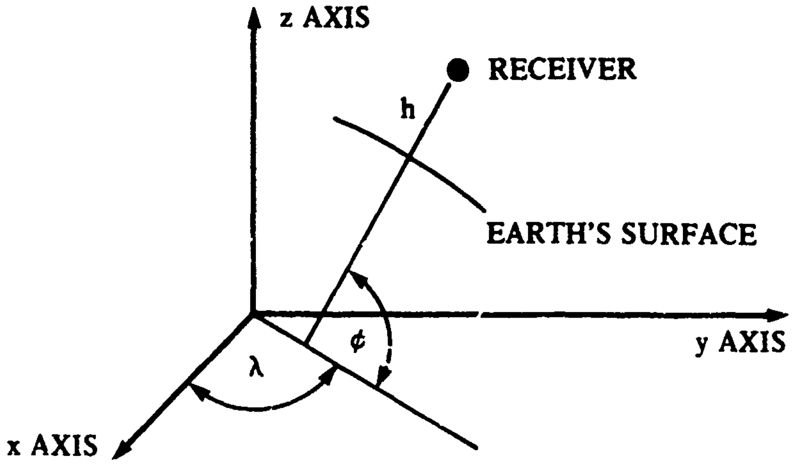

FIGURE 1. RECEIVER POSITION

If latitude and longitude are input in degrees, convert to radians.

$$\phi = \phi \cdot \pi / 180$$

$$\lambda = \lambda \cdot \pi / 180$$

Convert to ECEF cartesian coordinates.

$$x = \left[\frac{a_e}{\sqrt{i - e^2 \sin^2 \phi}} + h \right] \cos\phi \cos\lambda$$

$$y = \left[\frac{a_e}{\sqrt{i - e^2 \sin^2 \phi}} + h \right] \cos\phi \sin\lambda$$

$$z = \left[\frac{(i - e^2)\, a_e}{\sqrt{i - e^2 \sin^2 \phi}} + h \right] \sin\phi$$

DETERMINING SATELLITE QUADRANT AND ESTABLISHING WHETHER A SATELLITE IS HEALTHY AND IN VIEW[1]

The user inputs the following data

x_s, y_s, z_s	Satellite position in ECEF cartesian coordinates
x, y, z	Receiver position in ECEF cartesian coordinates
θ_{cut}	One cutoff angle for each of the four quadrants

The receiver is used as a reference point to define the quadrants as follows (see Appendix C for further explanation)

$0 \leqslant$ quadrant $1 < 90$ North-East

$90 \leqslant$ quadrant $2 < 180$ North-West

$180 \leqslant$ quadrant $3 < 270$ South-West

$270 \leqslant$ quadrant $4 < 360$ South-East

Establish the satellite quadrant, and define the vector from the receiver to the satellite, $\vec{P}$, by its three components P_x, P_y and P_z.

$$P_x = x_s - x$$

$$P_y = y_s - y$$

$$P_z = z_s - z$$

Define vectors $\vec{E}$ and $\vec{N}$, which with zenith, $\vec{Z}$, define a new coordinate system centered at the receiver.

$(E_x, E_y, E_z) = (-y, x, o)$ East

$(N_x, N_y, N_z) = (-xy, -yz, xy + y^2)$ North

Define the magnitudes.

$$|\vec{E}| = (E_x^2 + E_y^2 + E_z^2)^{1/2}$$

$$|\vec{N}| = (N_x^2 + N_y^2 + N_z^2)^{1/2}$$

Compute satellite position in the new $\vec{E} - \vec{N}$ plane.

$$\vec{P} \cdot \vec{E} = (-P_x \cdot y, P_y \cdot x, o)$$

$$\vec{P} \cdot \vec{N} = (-P_x \cdot x \cdot z, -P_y \cdot y \cdot z, P_z \cdot x \cdot y + P_z \cdot y^2)$$

$$(P_E, P_N) = \left(\frac{\vec{P} \cdot \vec{E}}{|\vec{E}|}, \frac{\vec{P} \cdot \vec{N}}{|\vec{N}|} \right)$$

P_E and P_N determine the satellite quadrant.

If $P_E = 0$ and $P_N = 0$ (exactly on azimuth, $\vec{Z}$) satellite is in quadrant 1

If $P_E > 0$ and $P_N > 0$ satellite is in quadrant 1

If $P_E \leqslant 0$ and $P_N > 0$ satellite is in quadrant 2

If $P_E < 0$ and $P_N \leqslant 0$ satellite is in quadrant 3

If $P_E \geqslant 0$ and $P_N < 0$ satellite is in quadrant 4

Establish whether or not the satellite is in view.

$$(P_x, P_y, P_z) = (x_s - x, y_s - y, z_s - z)$$

Compute the following, where $\vec{R}$ is the receiver position vector.

$$|\vec{P}| = (P_x^2 + P_y^2 + P_z^2)^{1/2}$$

$$|\vec{R}| = (x^2 + y^2 + z^2)^{1/2}$$

$$\vec{P} \cdot \vec{R} = P_x \cdot x + P_y \cdot y + P_z \cdot z$$

The angle between $\vec{P}$ and the horizon is given by β

If $\sin \beta < \sin \theta_{cut}$ the satellite is out of view

If $\sin \beta \geqslant \sin \theta_{cut}$ the satellite is in view

COMPUTING THE COVARIANCE MATRIXES FOR ALL GROUPS OF FOUR SATELLITES THAT ARE HEALTHY AND IN VIEW [1]

The user inputs the following data

x_s, y_s, z_s	Satellite positions for four satellites (and tracker number subscript, s)
x, y, z	Receiver position

For each of the four satellites compute a 4 x 4 matrix M_s.

First let

$$\Delta x_s = (x_s - x)$$

$$\Delta y_s = (y_s - y)$$

$$\Delta z_s = (z_s - z)$$

$$R_s = [\Delta x_s^2 + \Delta y_s^2 + \Delta z_s^2]^{1/2}$$

and

$$P1_s = \Delta x_s / R_s$$

$$P2_s = \Delta y_s / R_s$$

$$P3_s = \Delta z_s / R_s$$

$$P4_s = 1$$

Now,

$$M_s = \begin{bmatrix} P1_s \cdot P1_s & P1_s \cdot P2_s & P1_s \cdot P3_s & P1_s \cdot P4_s \\ P2_s \cdot P1_s & P2_s \cdot P2_s & P2_s \cdot P3_s & P2_s \cdot P4_s \\ P3_s \cdot P1_s & P3_s \cdot P2_s & P3_s \cdot P3_s & P3_s \cdot P4_s \\ P4_s \cdot P1_s & P4_s \cdot P2_s & P4_s \cdot P3_s & P4_s \cdot P4_s \end{bmatrix}$$

Let M_{total} be the sum of the four M_s

$$M_{total} = \sum_{s=1}^{4} M_s$$

The final step in this procedure is to compute the inverse of M_{total}. This is the covariance matrix. In the next section this is denoted by M^{-1}.

COMPUTING A DILUTION OF PRECISION IN POSITION FROM THE COVARIANCE MATRIX IN TERMS OF LATITUDE AND LONGITUDE [1]

The user inputs the following data

M^{-1}	Covariance matrix
ϕ, λ, h	Geodetic receiver position
e	Eccentricity
a_e	Semimajor axis length
x, y, z	Cartesian receiver position

The user receives the following data

$DOP_{\phi\lambda h}$	Dilution of precision in position, reflecting latitude, longitude, and height
$DOP_{\phi\lambda}$	Dilution of precision in position, reflecting latitude and longitude

Let M^{-1} be written as follows

$$M^{-1} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

M^{-1} was generated from variables in the ECEF cartesian (x, y, z) coordinate system. For users measuring position in this coordinate system, geometric dilution of precision is computed by

$$GDOP = Tr[M^{-1}] = m_{11} + m_{22} + m_{33} + m_{44}$$

The procedure described following these notes yields a dilution of precision suitable for users measuring position in the geodetic (ϕ, λ, h) coordinate system. Two dilution of precision equations are defined; one for users always at nearly constant height ($DOP_{\phi\lambda}$), and one for users whose height varies ($DOP_{\phi\lambda h}$).

Form a new matrix by "rotating" the upper left 3x3 matrix in M^{-1} into the geodetic coordinate system. Equivalently, multiply the upper left 3x3 matrix in M^{-1} by the following matrix of partials

$$\begin{bmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ \frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & \frac{\partial\lambda}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}$$

The following procedure shows the equations for the partials in the above matrix. Appendix D details how they are derived.

Compute the range in the x−y plane and the radius of curvature.

$$R = \sqrt{x^2 + y^2}$$

$$RADC = \frac{a_e (i - e^2)}{[i - e^2 \sin^2 \phi]^{3/2}} + h$$

Compute partials of geodetic latitude.

$$\frac{\partial\phi}{\partial x} = \frac{-\cos\lambda \sin\phi}{RADC} \qquad \frac{\partial\phi}{\partial y} = \frac{-\sin\lambda \sin\phi}{RADC} \qquad \frac{\partial\phi}{\partial z} = \frac{\cos\phi}{RADC}$$

Compute partials of longitude.

$$\frac{\partial\lambda}{\partial x} = \frac{-\sin\lambda}{R} \qquad \frac{\partial\lambda}{\partial y} = \frac{\cos\lambda}{R} \qquad \frac{\partial\lambda}{\partial z} = \text{zero}$$

Compute partials of height.

$$\frac{\partial h}{\partial x} = \cos\lambda\cos\phi \qquad \frac{\partial h}{\partial y} = \sin\lambda\cos\phi \qquad \frac{\partial h}{\partial z} = \sin\phi$$

Dilution of precision.

$$DOP_{\phi\lambda} = \frac{\partial\phi m_{11}}{\partial x} + \frac{\partial\phi m_{21}}{\partial y} + \frac{\partial\phi m_{31}}{\partial z} + \frac{\partial\lambda m_{12}}{\partial x} + \frac{\partial\lambda m_{22}}{\partial y} + \frac{\partial\lambda m_{32}}{\partial z}$$

$$DOP_{\phi\lambda h} = DOP_{\phi\lambda} + \frac{\partial h m_{31}}{\partial x} + \frac{\partial h m_{32}}{\partial y} + \frac{\partial h m_{33}}{\partial z}$$

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1. Stanley L. Meyerhoff, *GESAR Formulation and Software Working Papers*, Strategic Systems Department, Space Flight Sciences Branch, NSWC, Dahlgren, Virginia.

2. Roger R. Bate, Donald D. Mueller, and Jerry E. White, *Fundamentals of Astrodynamics*, Dover Publications, Inc., New York, 1971.

3. *NAVSTAR GPS Space Segment/Navigation Users Interfaces*, ICD-GPS-200, Rockwell International, Jan 1983.

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APPENDIX A
USING ALMANAC DATA TO COMPUTE SATELLITE POSITION

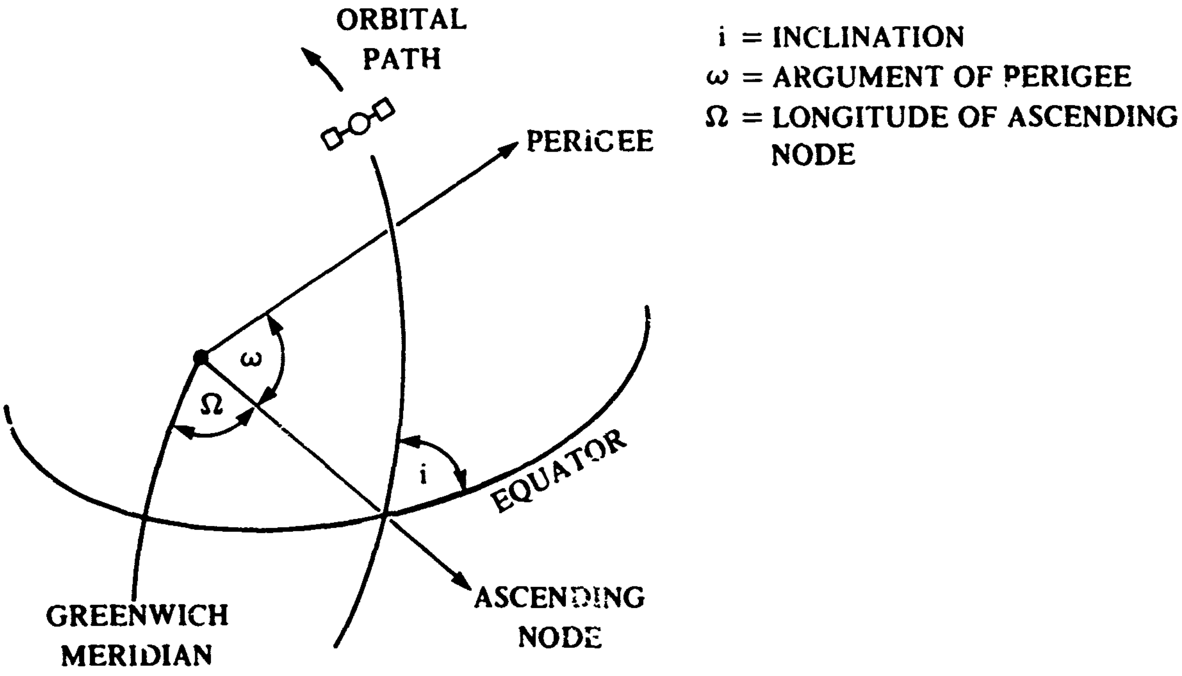

FIGURE A-1. CLASSICAL ORBITAL ELEMENTS

When computing satellite position from almanac data, first find the values of the sine and cosine of the true anomoly.[A-1]

$$\cos \nu = (\cos E - e)/(1 - e \cos E)$$

and

$$\sin \nu = \sqrt{1 - e^2} \cdot \sin E/(1 - e \cos E)$$

True anomoly is the angle shown in Figure A-2.

[A-1] R. R. Bate, D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics* (New York: Dover Publications, Inc., 1971), pp. 183-187.

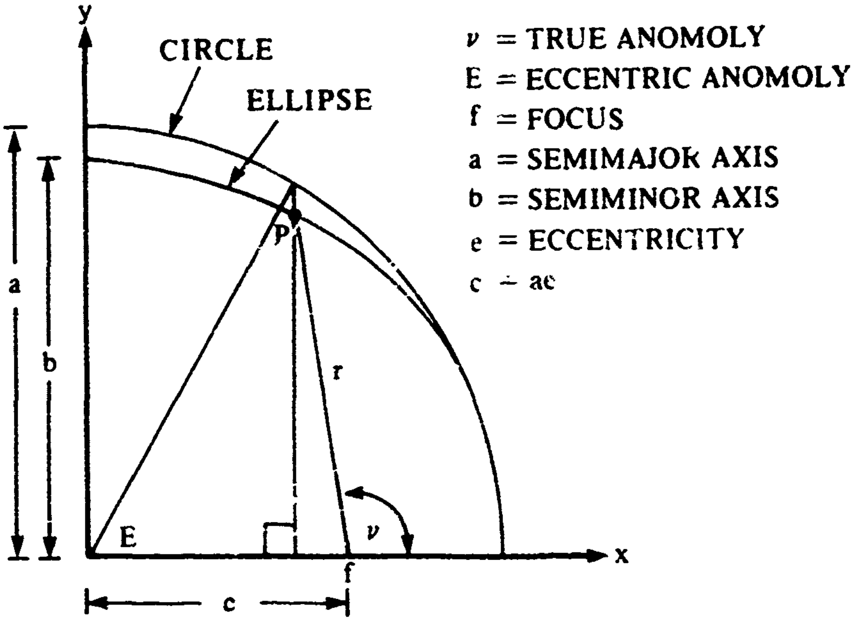

FIGURE A-2. TRUE ANOMOLY

Note two things

1. For any ellipse, like the one in Figure A-2, the focus is positioned such that

$$b^2 = a^2 - c^2$$

$$b^2 = a^2 - e^2 a^2$$

and

$$b = a\,(1 - e^2)^{1/2}$$

2. Given two points with the same x coordinate, one on the circle and one on the ellipse as drawn in Figure A-3, the ratio of y components is

$$\frac{y_{ellipse}}{y_{circle}} = \frac{b}{a}$$

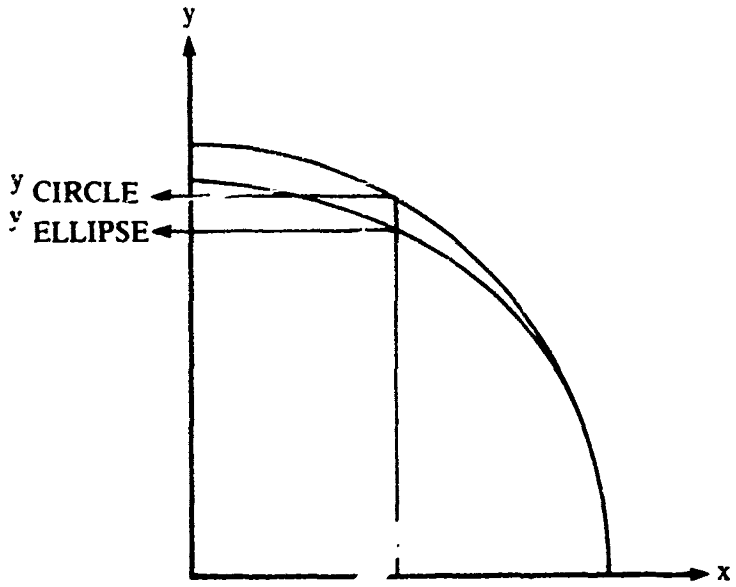

FIGURE A-3. ELLIPSE INSCRIBED WITHIN A CIRCLE

This can be seen from the equations for an ellipse and a circle. For a circle

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \qquad \text{and} \qquad y_{circle} = a\sqrt{1 - \frac{x^2}{a^2}}$$

For an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad y_{ellipse} = b\sqrt{1 - \frac{x^2}{a^2}}$$

Using Figure A-4, solve for r, then solve for true anomoly.

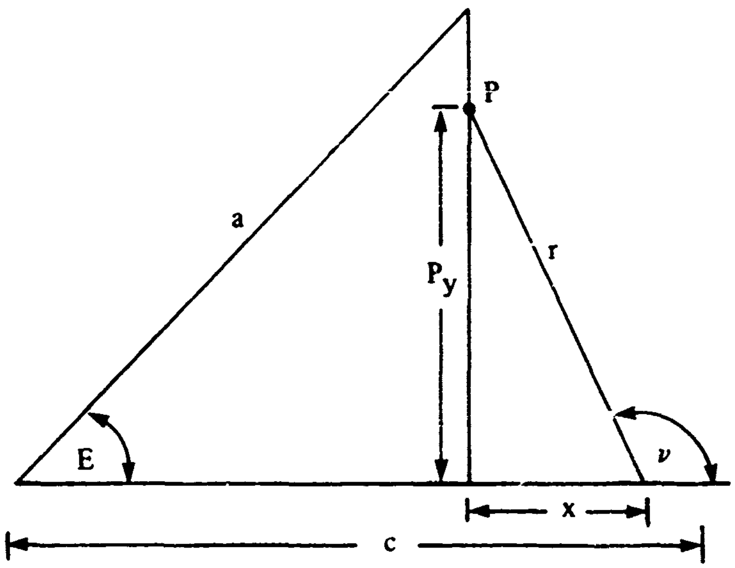

FIGURE A-4. TRUE ANOMOLY

$$r^2 = x^2 + P_y{}^2$$

$$r^2 = [c - a\cos E]^2 + [\left(\frac{b}{a}\right) \cdot a \sin E]^2$$

$$r^2 = a^2e^2 - 2a^2e\cos E + a^2\cos^2 E + b^2 \sin^2 E$$

$$r^2 = a^2\,[e^2 - 2e\cos E + \cos^2 E + (1 - e^2)\sin^2 E]$$

$$r^2 = a^2\,[e^2 - 2e\cos E - e^2 \sin^2 E + 1]$$

Use the expression $(1 - e\cos E)^2 = 1 - 2e\cos E + e^2\cos^2 E$

$$r^2 = a^2\,[e^2 + (1 - e\cos E)^2 - e^2\cos^2 E - e^2\sin^2 E]$$

so

$$r^2 = a^2\,(1 - e\cos E)^2$$

and

$$r = a\,(1 - e\cos E)$$

Finally

$$\cos\nu = -\sin\phi = -\frac{x}{r}$$

$$\cos\nu = -\frac{(c - a\cos E)}{a(1 - e\cos E)} = \frac{\cos E - e}{1 - e\cos E}$$

$$\sin\nu = \cos\phi = \frac{P_y}{r} = \frac{b\sin E}{r}$$

$$\sin\nu = \frac{\sqrt{1 - e^2}\cdot\sin E}{1 - e\cos E}$$

Using Figure A-5, define satellite position in the orbital plane.

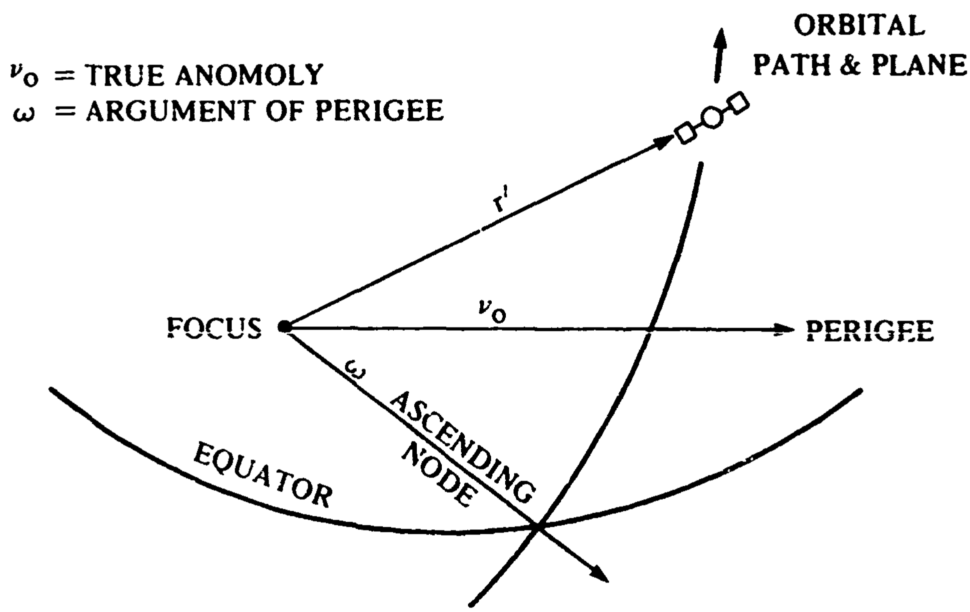

FIGURE A-5. SATELLITE POSITION IN THE ORBITAL PLANE

Define a coordinate system in the orbital plane where the x′ axis lies along the ascending node, and the y′ axis is perpendicular to the x′ axis with the origin at the focus. See Figure A-5.

Let

$$u_k = \nu_o + \omega$$

so that

$$r_{x'} = r' \cos u_k = r' \cos \nu_o \cos \omega - r' \sin \nu_o \sin \omega$$

$$r_{y'} = r' \sin u_k = r' \sin \nu_o \cos \omega + r' \sin \omega \cos \nu_o$$

Recall that

$$r' = a\,(1 - e \cos E)$$

The corrected longitude of tne ascending node is illustrated by Figure A-6.

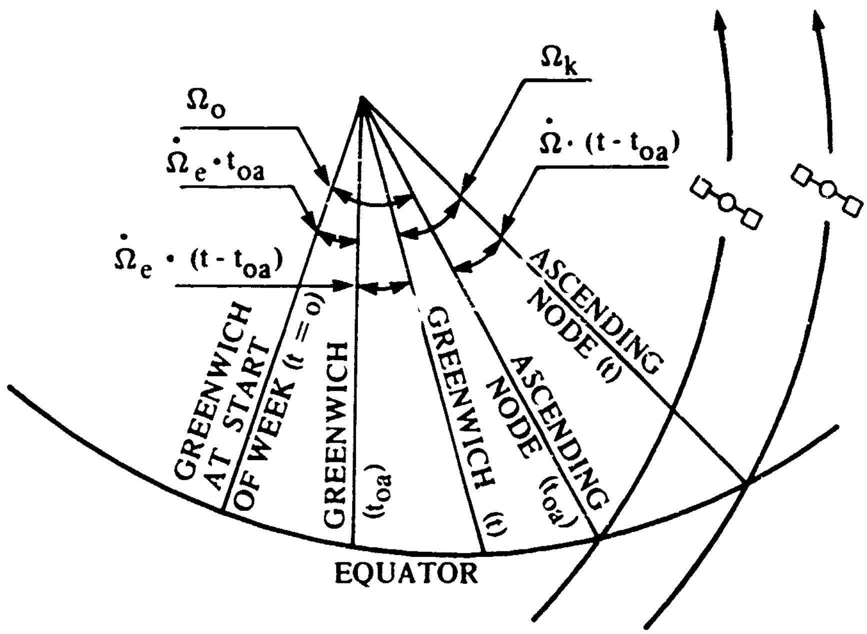

FIGURE A-6. CORRECTED LONGITUDE OF THE ASCENDING NODE

Ω_o = Initial right ascension, measured from Greenwich at the beginning of the week, to the ascending node at t_{oa}

$\dot{\Omega}$ = Rate of right ascension, the rate of orbital-plane drift around a fixed Earth

$\dot{\Omega}_e$ = Earth's rotation rate

t = Time at which satellite position is to be known

t_{oa} = Time of almanac data

Ω_k = Longitude of ascending node

From Figure A-6, Ω_k is given by

$$\Omega_k = \Omega_o + \dot{\Omega}\,(t - t_{oa}) - \dot{\Omega}_e\,(t - t_{oa})$$

$$\Omega_k = \Omega_o + \dot{\Omega}\,(t - t_{oa}) - \dot{\Omega}_e \cdot t$$

and where

$$t_k = t - t_{oa} + 604{,}800 \cdot (\#\text{ of weeks between } t \text{ and } t_{oa})$$

$$\Omega_k = \Omega_o + \dot{\Omega} \cdot t_k - \dot{\Omega}_e\,(t_k + t_{oa})$$

Figure A-7 illustrates the conversion from position in the orbital plane to position in ECEF cartesian coordinates.

$$r_x = r_{x'} \cos \Omega_k - r_{y'} \cos i_k \sin \Omega_k$$

$$r_y = r_{x'} \sin \Omega_k + r_{y'} \cos i_k \cos \Omega_k$$

$$r_z = r_{y'} \sin i_k$$

where $r_{x'}$ and $r_{y'}$ are the orbital plane coordinates, as defined previously.

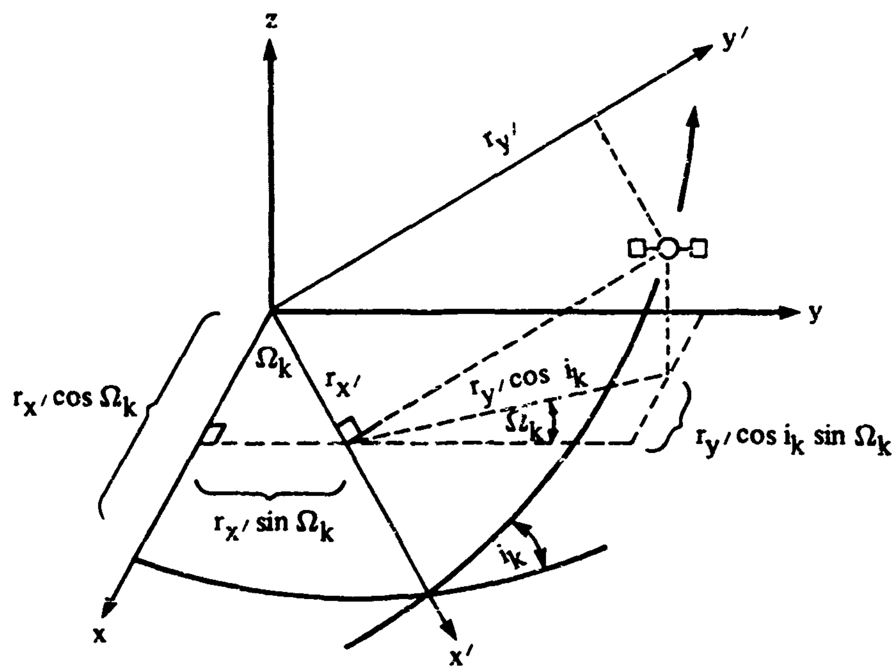

FIGURE A-7. ORBITAL PLANE AND ECEF CARTESIAN COORDINATES

APPENDIX B
CONVERTING GEODETIC COORDINATES TO ECEF CARTESIAN COORDINATES

Before converting geodetic coordinates to ECEF cartesian coordinates.[B-1] note two things

1. As seen in Figure B-1, for any ellipse, the focus is positioned such that

$$b^2 = a^2 - c^2$$

$$b^2 = a^2 - e^2 a^2$$

where

$$e = c / a$$

and

$$b = a(1 - e^2)^{1/2}$$

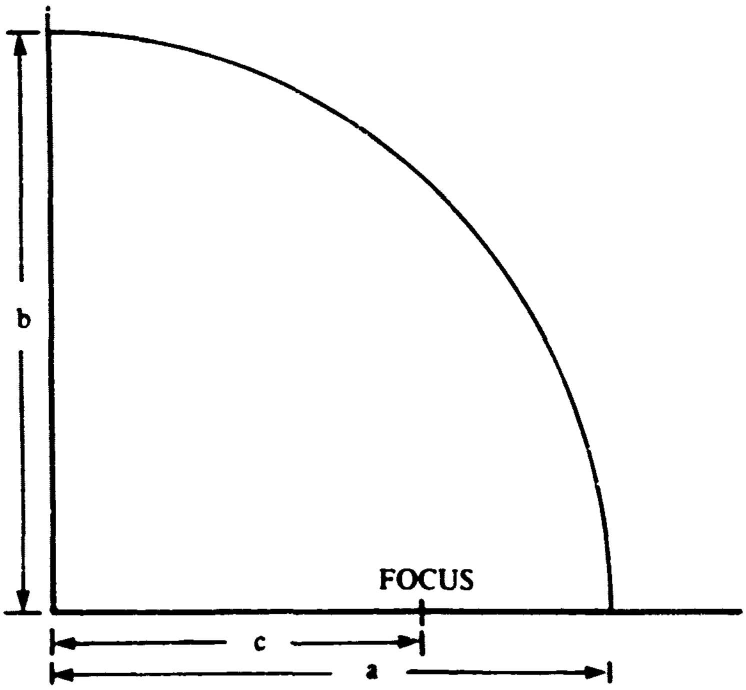

FIGURE B-1. ELLIPSE

B-1 R. R. Bate, D. D. Mueller, and J. F. White, *Fundamentals of Astrodynamics* (New York: Dover Publications, Inc., 1971), pp. 94-98

2. For a point on a circle and a point on an inscribed ellipse with the same x component, as seen in Figure B-2, the ratio of y components is

$$\frac{y_{ellipse}}{y_{circle}} = \frac{b}{a}$$

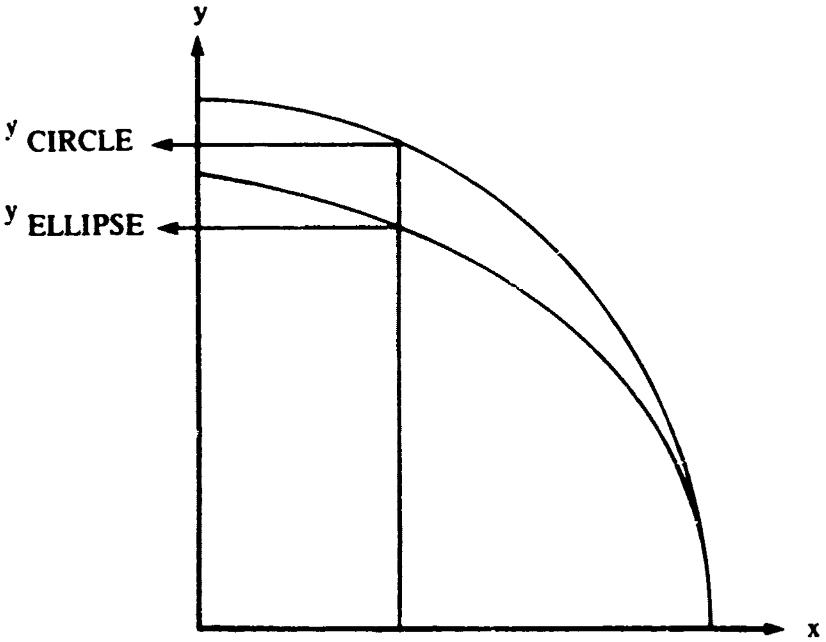

FIGURE B-2. ELLIPSE INSCRIBED WITHIN A CIRCLE

From Figure B-3 and the notes on the previous page

$$P_i = a_e \cos\beta$$

and

$$P_j = \left(\frac{b_e}{a_e}\right) a_e \sin\beta = a_e \sqrt{1 - e^2} \cdot \sin\beta$$

where these are to be redefined in terms of ϕ, not β.

j = AXIS FROM EARTH'S CENTER THROUGH THE NORTH POLE
i = AXIS IN EQUATORIAL PLANE THROUGH LONGITUDE OF P
a_e = SEMIMAJOR AXIS
b_e = SEMIMINOR AXIS
ϕ = GEODETIC LATITUDE
β = GEOCENTRIC LATITUDE
$\vec{t}$ = TANGENT TO EARTH'S SURFACE AT P
$\hat{n}$ = NORMAL TO $\vec{t}$, FORMS GEODETIC LATITUDE

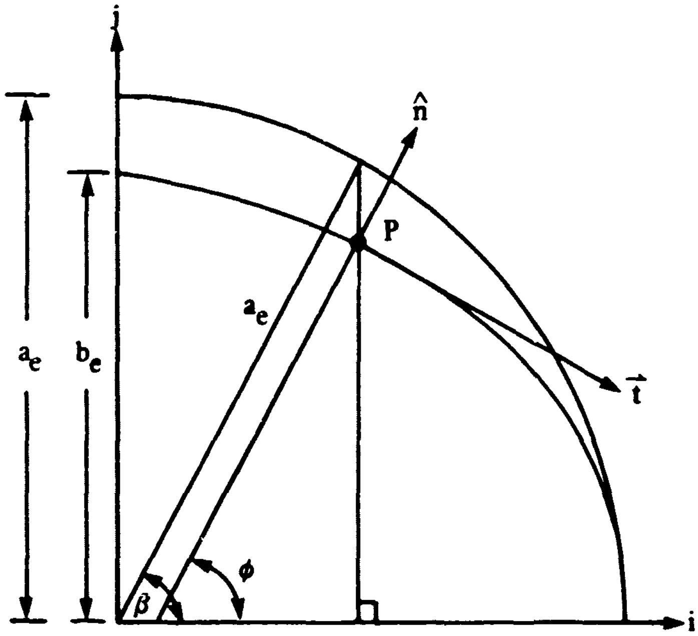

FIGURE B-3. CROSS SECTION OF EARTH INSCRIBED WITHIN A CIRCLE

The slope of $\vec{t}$ is $\dfrac{dj}{di}$ and the slope of $\hat{n}$ is $-\dfrac{di}{dj} = \tan\phi$

so differentiating P_i and P_j yields

$$\frac{dP_i}{dP_j} = \frac{a_e \sin\beta}{a_e\sqrt{1 - e^2}\cdot\cos\beta} = \frac{\tan\beta}{\sqrt{1 - e^2}} = \frac{\sin\phi}{\cos\phi}$$

Now find $\cos\beta$ and $\sin\beta$ in terms of ϕ.

Let

$$\tan\beta = \frac{\sqrt{1 - e^2}\cdot\sin\phi}{\cos\phi} = \frac{A}{B}$$

so that

$$\sin\beta = \frac{A}{\sqrt{A^2 + B^2}}$$

and

$$\cos\beta = \frac{B}{\sqrt{A^2 + B^2}}$$

$$\sin\beta = \frac{\sqrt{1 - e^2} \cdot \sin\phi}{\sqrt{1 - e^2 \sin^2\phi}}$$

$$\cos\beta = \frac{\cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}$$

and

$$P_i = \frac{a_e \cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}$$

$$P_j = \frac{a_e (1 - e^2) \sin\phi}{\sqrt{1 - e^2 \sin^2\phi}}$$

Let P' be a height h above the Earth's surface, as in Figure B-4. From this, it is seen that the coordinates of P' in the i-j frame are

$$P_i = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2\phi}} + h \right] \cos\phi$$

$$P_j = \left[\frac{a_e (1 - e^2)}{\sqrt{1 - e^2 \sin^2\phi}} + h \right] \sin\phi$$

Now place the i-j frame onto an x-y-z frame, with the j and z axes coinciding as in Figure B-5. The ECEF cartesian coordinates are

$$P_x = P_i \cos\lambda = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2\phi}} + h \right] \cos\phi \cos\lambda$$

$$P_y = P_i \sin\lambda = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2\phi}} + h \right] \cos\phi \sin\lambda$$

$$P_z = P_i = \left[\frac{(1 - e^2) a_e}{\sqrt{1 - e^2 \sin^2\phi}} + h \right] \sin\phi$$

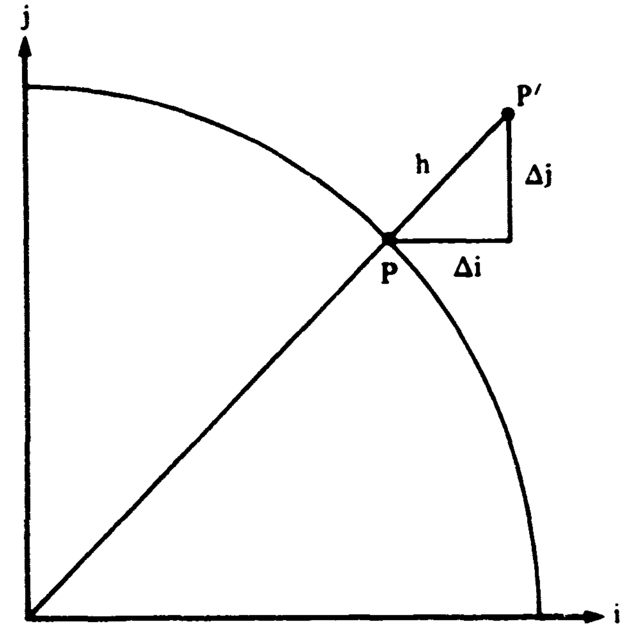

FIGURE B-4. ILLUSTRATION OF HEIGHT

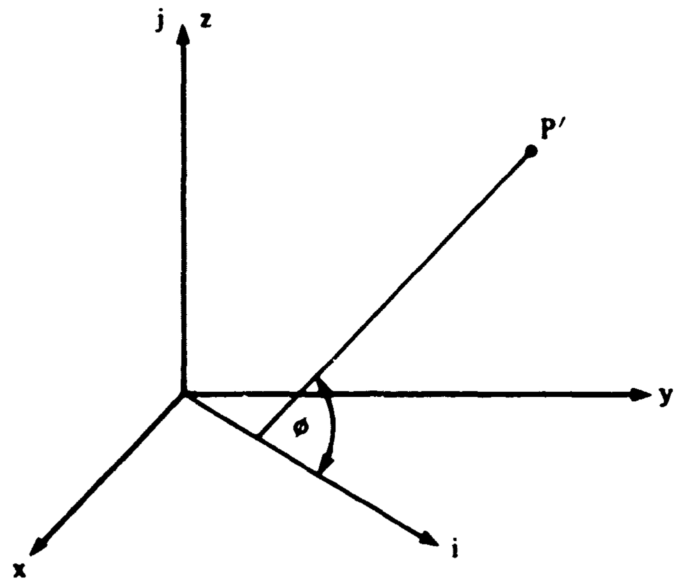

FIGURE B-5. ECEF CARTESIAN COORDINATES

APPENDIX C
DETERMINING SATELLITE QUADRANT AND WHETHER THE SATELLITE IS IN VIEW

Figure C-1 shows the Earth's northern hemisphere and the ECEF cartesian coordinate system. Receiver position is given by R; satellite position is given by S. (These data are necessary to determine the satellite quadrant and whether the satellite is in view.)[C-1]

$$\vec{P} = \vec{S} - \vec{R}$$

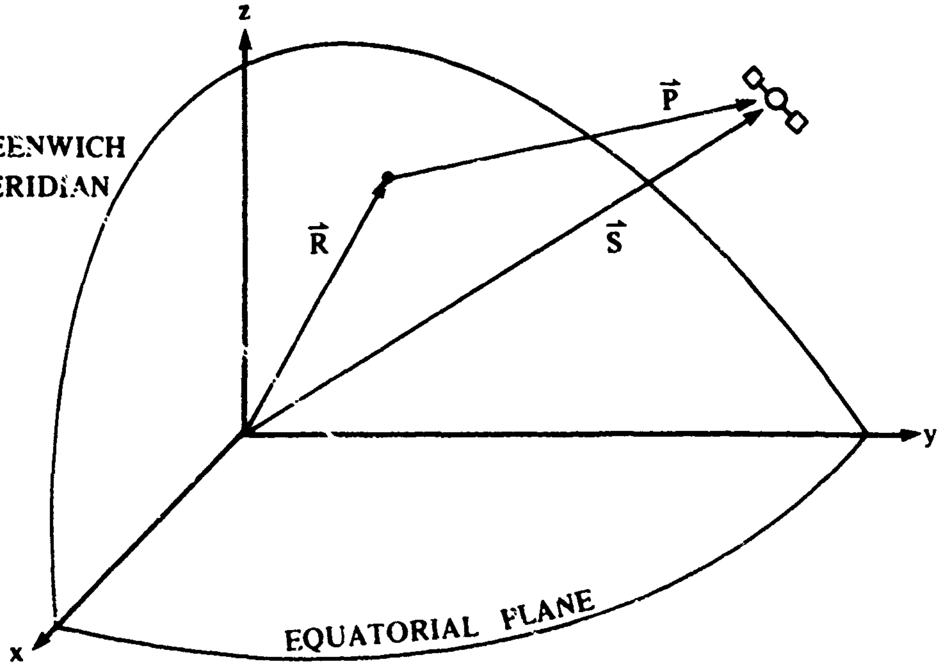

FIGURE C-1. NORTHERN HEMISPHERE AND ECEF CARTESIAN COORDINATES

Let $\hat{k}$ be the unit vector along the z axis. Define a new cartesian system, with its origin at the receiver, by taking the following cross products.

$$\vec{E} = \hat{k} \times \vec{R}$$

$$\vec{N} = \vec{R} \times \vec{E}$$

$$\vec{Z} = \vec{E} \times \vec{N}$$

C-1 S. I. Meyerhoff, *GESAR Formulation and Software Working Papers*, NSWC, Dahlgren, Virginia, Jan 1983

$\vec{E}$ is tangent to the Earth's surface at the receiver and points eastward.

$\vec{N}$ is nearly tangent to the surface at the receiver and points northward. If the Earth were a true sphere, $\vec{N}$ would be exactly tangent at the receiver.

$\vec{Z}$ is perpendicular to $\vec{E}$ and $\vec{N}$, and points toward the zenith.

The vectors $\vec{E}$, $\vec{N}$, and $\vec{Z}$ define the four quadrants, as shown in Figure C-2. The view is down along the $\vec{Z}$ axis. Let $\hat{e}$ and $\hat{n}$ be unit vectors in the east and north directions. Thus, satellite position relative to the receiver, $\vec{P}$, is given in this system by

$$\vec{P}_{enz} = |\vec{P}| \cos\beta\, \hat{e} + |\vec{P}| \cos\alpha\, \hat{n}$$

$$\vec{P}_{enz} = \frac{|\vec{P}|\,[\vec{P}\cdot\vec{E}]\,\hat{e}}{|\vec{P}|\,|\vec{E}|} + \frac{|\vec{P}|\,[\vec{P}\cdot\vec{N}]\,\hat{n}}{|\vec{P}|\,|\vec{N}|}$$

$$\vec{P}_{enz} = \frac{\vec{P}\cdot\vec{E}\,\hat{e}}{|\vec{E}|} + \frac{\vec{P}\cdot\vec{N}\,\hat{n}}{|\vec{N}|}$$

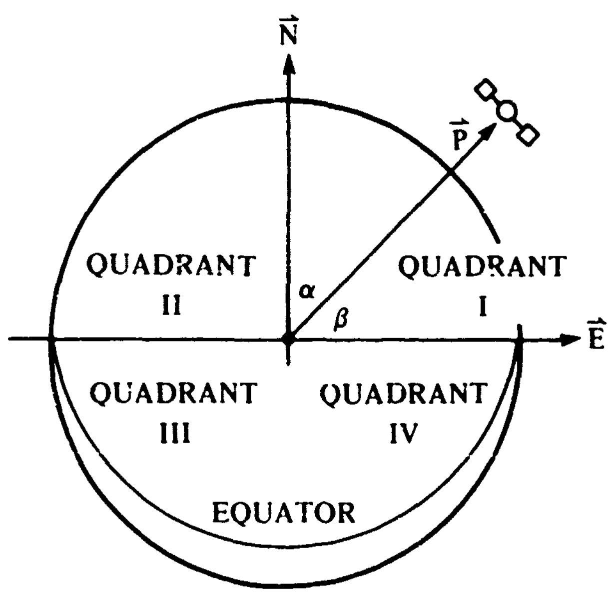

FIGURE C-2. ILLUSTRATION OF QUADRANTS

Define

$$\vec{P}_{enz} \equiv (P_E, P_N)$$

By examining P_E and P_N, (with predetermined boundary conditions), the quadrant in which the satellite is located can be determined. Each quadrant has a certain predetermined cutoff angle, θ_{cut}, measured from the horizon. If the angle between the horizon and the vector from the receiver to the satellite is less than θ_{cut}, the receiver is deemed to be out of view.

To determine whether the satellite is in view, find β, as illustrated in Figure C-3, and see if it is less than or greater than θ_{cut}.

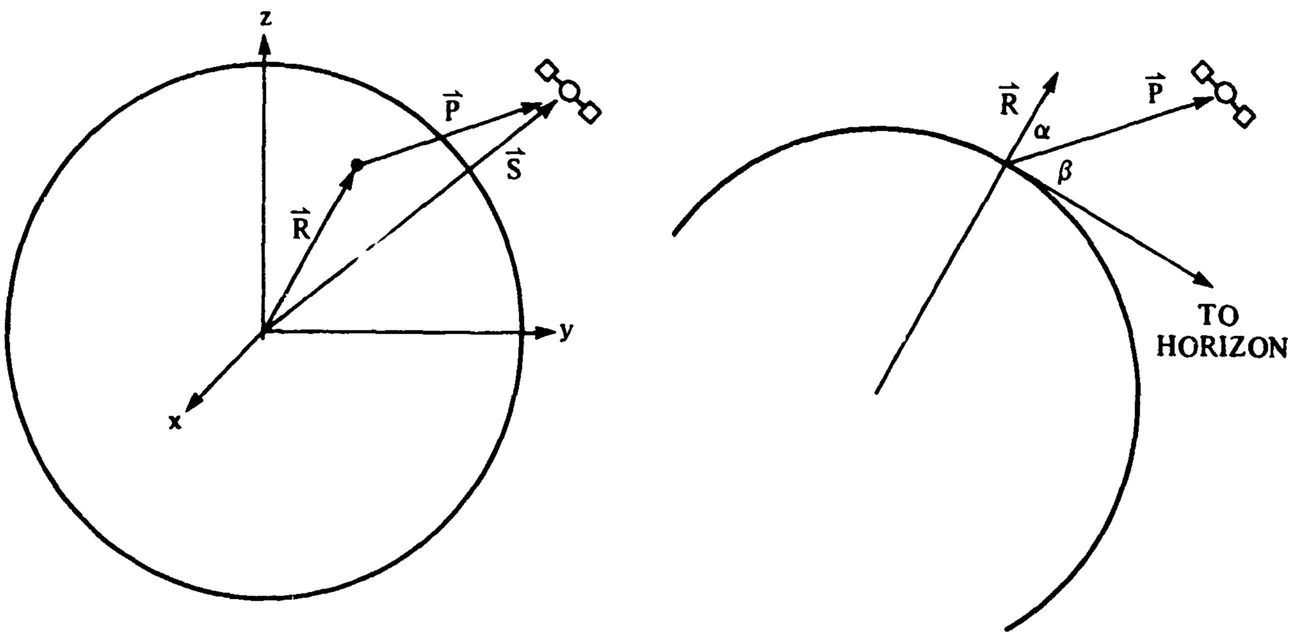

FIGURE C-3. TWO VIEWS OF SATELLITE POSITION

Where $\vec{R}$ and $\vec{P}$ are both known

$$\sin\beta = \cos\alpha = \frac{\vec{R}\cdot\vec{P}}{|\vec{R}|\,|\vec{P}|}$$

If

$\sin\beta < \sin\theta_{cut}$ the satellite is out of view.

If

$\sin\beta \geqslant \sin\theta_{cut}$ the satellite is in view.

APPENDIX D
DIFFERENTIATING EXPRESSIONS FOR LATITUDE, LONGITUDE, AND HEIGHT WITH RESPECT TO X, Y, AND Z

Finding expressions for geodetic latitude, longitude, and height; and differentiating them with respect to x, y, and z requires finding an expression for tan ϕ.[D-1]

As shown in Appendix B

$$z = \left[\frac{(1 - e^2)\, a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h \right] \sin \phi_1$$

and as shown in Appendix E

$$\tan \phi_1 = \frac{\tan \phi_2}{1 - e^2}$$

Using Figures D-1, D-2, and the relations that follow, find expressions for $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, and $\frac{\partial \phi}{\partial z}$.

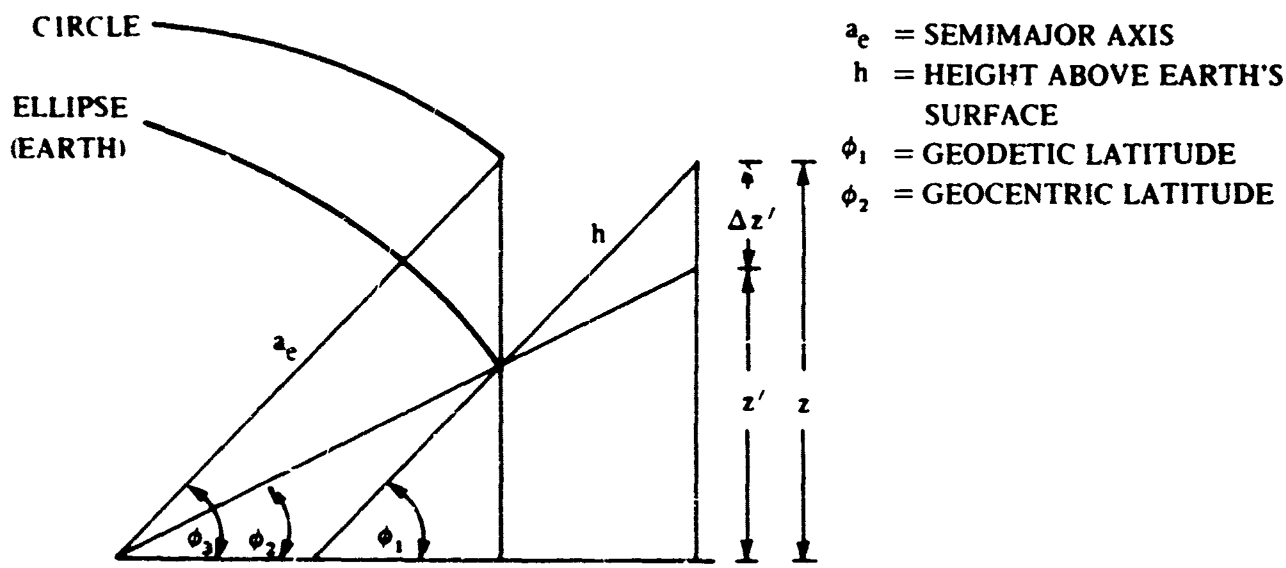

FIGURE D-1. GEODETIC AND GEOCENTRIC LATITUDES

[D-1] S. L. Meyerhoff, *GESAR Formulation and Software Working Papers*, NSWC, Dahlgren, Virginia, Jan 1983.

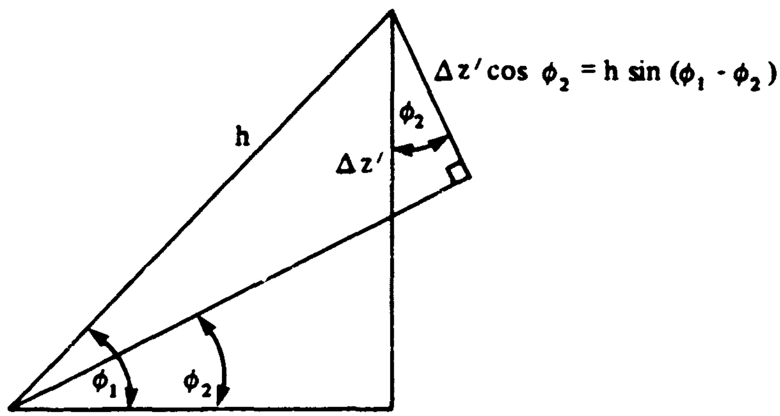

FIGURE D-2. HEIGHT AND GEODETIC AND GEOCENTRIC LATITUDES

$$\tan\phi_1 = \frac{\tan\varphi_2}{1-e^2} = \frac{z'}{R}\cdot\frac{1}{(1-e^2)} = \frac{z-\Delta z}{R}\cdot\frac{1}{(1-e^2)}$$

and

$$\Delta z' = \frac{h\sin(\phi_1-\phi_2)}{\cos\phi_2} = h\sin\phi_1 - h\tan\phi_2\cos\phi_1$$

$$\Delta z' = h\sin\phi_1 - h\cdot(1-e^2)\sin\phi_1$$

$$\Delta z' = e^2\cdot h\cdot\sin\phi_1$$

so

$$\tan\phi_1 = \frac{z}{R(1-e^2)} - \frac{\Delta z'}{R(1-e^2)} = \frac{a_e\sin\phi_1}{R\sqrt{1-e^2\sin^2\phi_1}} + \frac{h\sin\phi_1}{R(1-e^2)} - \frac{h e^2\sin\phi_1}{R(1-e^2)}$$

$$\tan\phi_1 = \left[\frac{a_e}{\sqrt{1-e^2\sin^2\phi_1}} + h\right]\frac{\sin\phi_1}{R}$$

Note that an equivalent expression is

$$\tan \phi_1 = \frac{z}{R} + \frac{a_e e^2 \sin \phi_1}{R\sqrt{1 - e^2 \sin^2 \phi_1}}$$

From this expression, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, and $\frac{\partial \phi}{\partial z}$ can be found as shown on the following pages.

The above equation for latitude has the problem of containing ϕ_1, on both sides. To find a numerical value for ϕ_1, when z, e, a and $R = (x^2 + y^2)^{1/2}$ are known, first estimate ϕ_1, by

$$\phi_1 = \tan^{-1}\left(\frac{z}{R}\right)$$

Insert this into the right hand side, reevaluate tan ϕ_1, and repeat until negligible change occurs between iterations.

Find $\frac{\partial \phi_1}{\partial x}$

Use the following

$$R = \sqrt{x^2 + y^2} = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h\right] \cos \phi_1$$

$$\frac{\partial}{\partial x}\left(\frac{1}{R}\right) = -\frac{x}{R^3} \qquad \frac{\partial}{\partial x}(1 - e^2 \sin^2 \phi_1)^{-1/2} = \frac{e^2 \sin \phi_1 \cos \phi_1}{(1 - e^2 \sin^2 \phi_1)^{3/2}} \cdot \frac{d\phi_1}{dx}$$

Factor 1/R out of the expression for tan ϕ_1, and solve from there

$$\tan \phi_1 = \frac{z}{R} + \frac{a_e e^2 \sin \phi_1}{R\sqrt{1 - e^2 \sin^2 \phi_1}}$$

$$\tan \phi_1 = \frac{1}{R}\left[z + \frac{a_e \cdot e^2 \sin \phi_1}{\sqrt{1 - e^2 \sin^2 \phi_1}}\right]$$

Differentiate

$$\sec^2 \phi_1 \frac{d\phi_1}{dx} = -\frac{x}{R^3}\left[z + \frac{a_e e^2 \sin \phi_1}{\sqrt{1 - e^2 \sin^2 \phi_1}}\right] + \frac{1}{R}\left[\frac{a_e e^4 \sin^2 \phi_1 \cos \phi_1}{(1 - e^2 \sin^2 \phi_1)^{3/2}}\right.$$

$$\left. + \frac{a_e e^2 \cos \phi_1}{\sqrt{1 - e^2 \sin^2 \phi}}\right] \frac{d\phi_1}{dx}$$

$$\sec^2 \phi_1 \frac{d\phi_1}{dx} = -\frac{x}{R^2} \tan \phi_1 + \frac{1}{R} \frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} \cdot e^2 \cos \phi_1$$

$$\left[\frac{e^2 \sin^2 \phi_1}{1 - e^2 \sin^2 \phi_1} + 1\right] \frac{d\phi_1}{dx}$$

$$-x \tan \phi_1 = \left\{ R^2 \sec^2 \phi_1 - \frac{R e^2 \cos \phi_1 \, a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} \left[\frac{1}{1 - e^2 \sin^2 \phi_1}\right]\right\} \frac{\partial \phi_1}{\partial x}$$

$$-x \tan \phi_1 = \left\{\left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h\right]^2 \frac{\cos^2 \phi_1}{\cos^2 \phi_1} - \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h\right]\right.$$

$$\left. \cdot \frac{e^2 \cos^2 \phi_1}{\sqrt{1 - e^2 \sin^2 \phi_1}} \cdot a_e \cdot \frac{1}{[1 - e^2 \sin^2 \phi_1]}\right\} \frac{d\phi_1}{dx}$$

$$-x \tan \phi_1 = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h\right] \left\{\left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h\right]\right.$$

$$\left. - \frac{a_e e^2 \cos^2 \phi_1}{\sqrt{1 - e^2 \sin^2 \phi_1}} \cdot \left[\frac{1}{1 - e^2 \sin^2 \phi_1}\right]\right\} \frac{d\phi_1}{dx}$$

$$-x \tan \phi_1 = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h \right] \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} \left[1 - \frac{e^2 \cos^2 \phi_1}{1 - e^2 \sin^2 \phi_1} \right] \right.$$

$$\left. + h \right] \frac{d\phi_1}{dx}$$

$$-x \tan \phi_1 = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h \right] \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} \left[\frac{1 - e^2}{1 - e^2 \sin^2 \phi_1} \right] \right.$$

$$\left. + h \right] \frac{d\phi_1}{dx}$$

Since $x = R \cos \lambda$ and $R = \left[\dfrac{a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h \right] \cos \phi_1$

and letting the radius of curvature be

$$RADC = \frac{(1 \quad e^2)\, a_e}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h$$

It follows that

$$-x \tan \phi_1 = -R \cos \lambda \tan \phi_1$$

$$-R \cos \lambda \tan \phi_1 = \frac{R}{\cos \phi_1} \cdot RADC \cdot \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = \frac{-\cos \lambda \sin \phi_1}{RADC}$$

Similarly, for $\frac{\partial \phi_1}{\partial y}$, the last steps are

$$-y \tan \phi_1 = -R \sin \lambda \tan \phi_1$$

$$-R \sin \lambda \tan \phi_1 = \frac{R}{\cos \phi_1} \cdot \text{RADC} \quad \frac{\partial \phi_1}{\partial y}$$

$$\frac{\partial \phi_1}{\partial y} = -\frac{\sin \lambda \sin \phi_1}{\text{RADC}}$$

Now find $\frac{\partial \phi_1}{\partial z}$, (let $\phi = \phi_1$)

$$\tan \phi_1 = \frac{1}{R}\left[z + \frac{a_e \cdot e^2 \cdot \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \right]$$

as before, but $R = \sqrt{x^2 + y^2} = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi}} + h \right] \cos \phi$

is now constant

Differentiate

$$\sec^2 \phi \frac{\partial \phi}{\partial z} = \frac{1}{R} \left[\frac{\partial z}{\partial z} + \frac{a_e \cdot e^2 \sin^2 \phi \cos \phi}{(1 - e^2 \sin^2 \phi)^{3/2}} \cdot \frac{\partial \phi}{\partial z} + \frac{a_e e^2 \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \cdot \frac{\partial \phi}{\partial z} \right]$$

$$\sec^2 \phi \frac{\partial \phi}{\partial z} = \frac{1}{R} \left[1 + \frac{a_e e^2 \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \left[\frac{e^2 \sin^2 \phi}{(1 - e^2 \sin^2 \phi)} + 1 \right] \frac{\partial \phi}{\partial z} \right]$$

$$\sec^2 \phi \frac{\partial \phi}{\partial z} = \frac{1}{R} \left[1 + \frac{a_e e^2 \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \cdot \frac{1}{(1 - e^2 \sin^2 \phi)} \cdot \frac{\partial \phi}{\partial z} \right]$$

$$\left[\frac{a_e}{\sqrt{1-e^2\sin^2\phi}}+h\right]\frac{\cos\phi}{\cos^2\phi}\quad\frac{\partial\phi}{\partial z}=1+\frac{a_e\,e^2\cos\phi}{\sqrt{1-e^2\sin^2\phi}}\cdot\frac{1}{(1-e^2\sin^2\phi)}\cdot\frac{d\phi}{dz}$$

$$\frac{a_e}{\sqrt{1-e^2\sin^2\phi}}\quad\frac{\partial\phi}{\partial z}+h\,\frac{\partial\phi}{\partial z}=\cos\phi+\frac{a_e\,e^2\cos^2\phi}{\sqrt{1-e^2\sin^2\phi}}\cdot\frac{1}{(1-e^2\sin^2\phi)}\cdot\frac{d\phi}{dz}$$

$$\frac{a_e}{\sqrt{1-e^2\sin^2\phi}}\left[1-\frac{e^2\cos^2\phi}{1-e^2\sin^2\phi}\right]\frac{\partial\phi}{\partial z}+h\,\frac{\partial\phi}{\partial z}=\cos\phi$$

$$\left[\frac{(1-e^2)\,a_e}{\sqrt{1-e^2\sin^2\phi}}\cdot\frac{1}{(1-e^2\sin^2\phi)}+h\right]\frac{\partial\phi}{\partial z}=\cos\phi$$

Thus $\dfrac{\partial\phi}{\partial z}=\dfrac{\cos\phi}{RADC}$

where again RADC is the radius of curvature

$$RADC=\frac{(1-e^2)\,a_e}{(1-e^2\sin^2\phi)^{3/2}}+h$$

Find $\dfrac{\partial\lambda}{\partial x}$, $\dfrac{\partial\lambda}{\partial y}$, and $\dfrac{\partial\lambda}{\partial z}$

Since R and the angle λ swept out by R are in the x-y plane.

$$\frac{\partial\lambda}{\partial z}=0$$

Now find $\dfrac{\partial\lambda}{\partial x}$ and $\dfrac{\partial\lambda}{\partial y}$

$$\cos\lambda=\frac{x}{R}=(x^{-2})^{-1/2}\cdot(x^2+y^2)^{-1/2}=(1+y^2x^{-2})^{-1/2}$$

After differentiating:

$$-\sin\lambda \frac{\partial\lambda}{\partial x} = -\frac{1}{2}(1 + y^2 x^{-2})^{-3/2}(-2x^{-3} y^2)$$

$$-\sin\lambda \frac{\partial\lambda}{\partial x} = (1 + y^2 x^{-2})^{-3/2}(x^2)^{-3/2}\ y^2$$

$$-\sin\lambda \frac{\partial\lambda}{\partial x} = (x^2 + y^2)^{-3/2} y^2$$

$$-\sin\lambda \frac{\partial\lambda}{\partial x} = \frac{y^2}{R^3}$$

$$\frac{\partial\lambda}{\partial x} = -\frac{R}{y}\cdot\frac{y^2}{R^3} = -\frac{y}{R^2}$$

$$\frac{\partial\lambda}{\partial x} = -\frac{\sin\lambda}{R}$$

Similarly

$$\sin\lambda = \frac{y}{R} = (x^2 y^{-2} + 1)^{-1/2}$$

$$\cos\lambda \frac{\partial\lambda}{\partial y} = (x^2 + y^2)^{-3/2}\cdot x^2$$

$$\frac{\partial\lambda}{\partial y} = \frac{x}{R^2}$$

$$\frac{\partial\lambda}{\partial y} = \frac{\cos\lambda}{R}$$

Find $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial y}$, **and** $\frac{\partial h}{\partial z}$

Use $R = \left[\dfrac{a_e}{\sqrt{1 - e^2 \sin^2 \phi}} + h\right] \cos\phi \qquad \dfrac{\partial\phi}{\partial x} = \dfrac{-\cos\lambda \sin\phi}{RADC} \qquad x = R\cos\lambda$

and $RADC = \dfrac{(1 - e^2) a_e}{(1 - e^2 \sin^2 \phi)^{3/2} + h}$

Start with

$$h = \frac{-a_e}{\sqrt{1 - e^2 \sin^2 \phi}} + \frac{R}{\cos\phi}$$

and differentiate

$$\frac{\partial h}{\partial x} = \frac{-a_e \cdot e^2 \sin\phi\cos\phi}{(1 - e^2 \sin^2 \phi)^{3/2}} \frac{\partial\phi}{\partial x} + \frac{x}{R} \cdot \frac{1}{\cos\phi} + \frac{R\sin\phi}{\cos^2\phi} \frac{\partial\phi}{\partial x}$$

$$\frac{\partial h}{\partial x} = \frac{a_e \cdot e^2 \sin^2\phi\cos\phi\cos\lambda}{RADC \cdot (1 - e^2 \sin^2 \phi)^{3/2}} + \frac{\cos\lambda}{\cos\phi} - \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi}} + h\right] \frac{\sin^2\phi\cos\lambda}{RADC \cdot \cos\phi}$$

$$\frac{\partial h}{\partial x} = \frac{\cos\lambda}{\cos\phi} \left[\frac{a_e e^2 \sin^2\phi\cos^2\phi}{RADC \cdot (1 - e^2 \sin^2 \phi)^{3/2}} - \frac{a_e \sin^2\phi}{\sqrt{1 - e^2 \sin^2 \phi}} \cdot \frac{1}{RADC} - \frac{h \sin^2\phi}{RADC} + 1\right]$$

$$\frac{\partial h}{\partial x} = \frac{\cos\lambda}{\cos\phi} \left[\frac{\sin^2\phi}{RADC} \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 \phi}} \left[\frac{e^2\cos^2\phi}{(1 - e^2 \sin^2 \phi)} - 1\right] - h\right] + 1\right]$$

$$\frac{\partial h}{\partial x} = \frac{\cos\lambda}{\cos\phi} \left[\frac{\sin^2\phi}{RADC} \left[\frac{(e^2 - 1) a_e}{\sqrt{1 - e^2 \sin^2 \phi}} \cdot \frac{1}{(1 - e^2 \sin^2 \phi)} - h\right] + 1\right]$$

$$\frac{\partial h}{\partial x} = \frac{\cos\lambda}{\cos\phi} \left[\frac{\sin^2\phi}{RADC} \cdot RADC + 1\right]$$

$$\frac{\partial h}{\partial x} = \cos\lambda\cos\phi$$

Similarly $\dfrac{\partial h}{\partial y} = \sin\lambda\cos\phi$

Now find $\dfrac{\partial h}{\partial z}$, where as before

$$h = \frac{-a_e}{\sqrt{1 - e^2 \sin^2\phi}} + \frac{R}{\cos\phi} \quad \text{and} \quad \frac{\partial\phi}{\partial z} = \frac{\cos\phi}{\text{RADC}}$$

So

$$\frac{\partial h}{\partial z} = \frac{-a_e\, e^2 \sin\phi\cos\phi}{(1 - e^2 \sin^2\phi)^{3/2}} \frac{\partial\phi}{\partial z} + \frac{R\sin\phi}{\cos^2\phi} \frac{\partial\phi}{\partial z}$$

$$\frac{\partial h}{\partial z} = \frac{-a_e \cdot e^2 \sin\phi\cos^2\phi}{\text{RADC} \cdot (1 - e^2 \sin^2\phi)^{3/2}} + \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2\phi}} + h\right]\frac{\sin\phi}{\text{RADC}}$$

$$\frac{\partial h}{\partial z} = \frac{\sin\phi}{\text{RADC}}\left[\frac{a_e}{\sqrt{1 - e^2 \sin^2\phi}}\left[1 - \frac{e^2\cos^2\phi}{(1 - e^2 \sin^2\phi)}\right] + h\right]$$

$$\frac{\partial h}{\partial z} = \frac{\sin\phi}{\text{RADC}}\left[\frac{(1 - e^2)\, a_e}{\sqrt{1 - e^2 \sin^2\phi}} \cdot \frac{1}{(1 - e^2 \sin^2\phi)}\right] = \frac{\sin\phi}{\text{RADC}} \cdot \text{RADC}$$

and finally

$$\frac{\partial h}{\partial z} = \sin\phi$$

APPENDIX E
THE RELATIONSHIP BETWEEN GEODETIC AND GEOCENTRIC LATITUDE

The relationship between geodetic and geocentric latitude [E-1] is shown in Figure E-1. Assuming that the Earth has the shape of an ellipsoid, show that

$$\tan \phi_1 = \frac{\tan \phi_2}{(1 - e^2)}$$

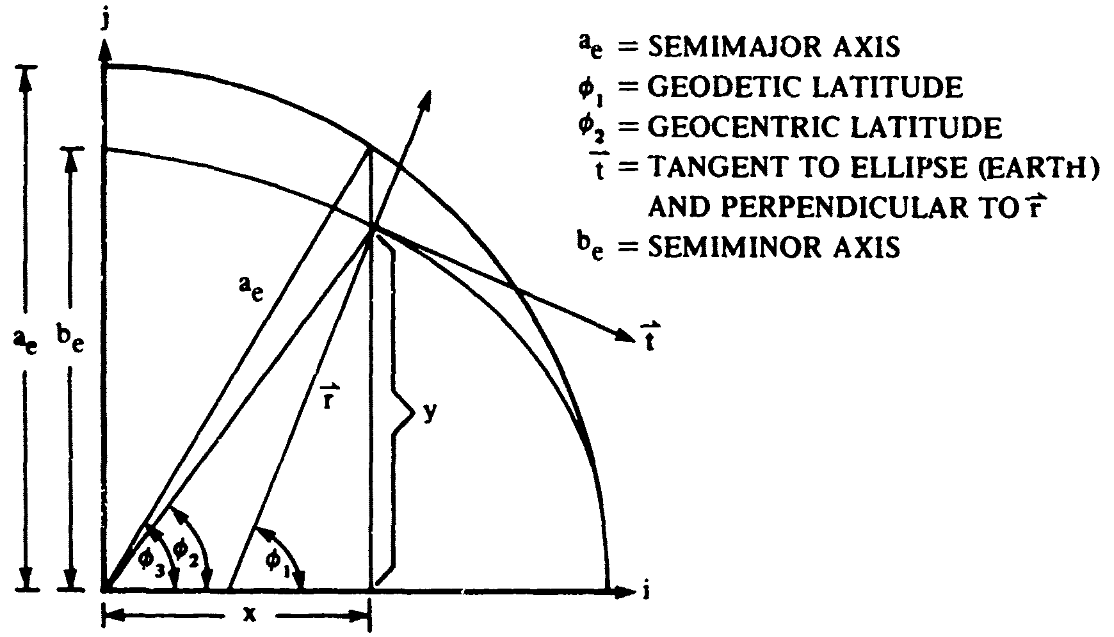

FIGURE E-1. RELATIONSHIP BETWEEN GEODETIC AND GEOCENTRIC LATITUDES

As shown in Appendix A, where the x components are the same, the y components of a point on a circle and a point on an inscribed ellipse have the following ratio

$$\frac{y_{ellipse}}{y_{circle}} = \frac{b}{a}$$

E-1 R. R. Bate, D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics* (New York: Dover Publications, Inc., 1971), pp. 96-97

where $b \quad = \quad a\sqrt{1 - e^2}$

Thus, the angles ϕ_2 and ϕ_3 are related by

$$\tan \phi_2 \quad = \quad \sqrt{1 - e^2}\ \tan \phi_3$$

The slope of tangent vector $\vec{t}$ is $\dfrac{\partial y}{\partial x}$, and the slope of its normal $\vec{r}$ is $-\dfrac{\partial x}{\partial y}$

From Figure E-1

$$x = a_e \cos \phi_3 \qquad\qquad y = \frac{b_e}{a_e} \cdot a_e \sin \phi_3$$

Thus

$$\tan \phi_1 \quad = \quad -\frac{\partial x}{\partial y} \quad = \quad \frac{a_e \sin \phi_3}{a_e \sqrt{1 - e^2}\ \cos \phi_3} = \frac{\tan \phi_3}{\sqrt{1 - e^2}}$$

and

$$\tan \phi_1 \quad = \quad \frac{\tan \phi_2}{1 - e^2}$$

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